Elimination

Elimination matrices differ from the identity by a single element. The elimination matrix E_{ij} has $-l$ in the i, j position.

This matrix is used to produce a zero in the i, j position. Take $E_{21} = \begin{pmatrix} 1 & & \\ 1 & 1 & \\ & & 1 \end{pmatrix}$. We begin with this and proceed to obtain an upper triangular matrix via other elimination matrices:

$$\begin{pmatrix} 1 & & \\ 1 & 1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ -1 & 0 & 1 \\ 2 & -5 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 2 & 4 \\ 2 & -5 & 4 \end{pmatrix}$$

$$\begin{pmatrix} 1 & & \\ & 1 & \\ -2 & & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 2 & 4 \\ 2 & -5 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 2 & 4 \\ 0 & -9 & -2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & & \\ & 1 & \\ & \frac{9}{2} & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 2 & 4 \\ 0 & -9 & -2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 2 & 4 \\ 0 & 0 & 16 \end{pmatrix}$$

Row Exchange

Sometimes one needs to exchange two rows in a matrix. To do this, we exchange rows of the identity appropriately. To exchange row 2 and row 3, we use

$$P_{23} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

We now note that this does what it's suppose to do

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ -1 & 0 & 1 \\ 2 & -5 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & -5 & 4 \\ -1 & 0 & 1 \end{pmatrix}$$

The first row of P_{23} has a 1 in the first entry. This selects the first row of the other matrix. The second row has a 1 in the third entry. This selects the 3rd row. The third row of P_{23} has a 1 in the 2nd entry selecting the 2nd row.

Exercise

Start with the four equations

$$-x_{i+1} + 2x_i - x_{i-1} = i \qquad i = 1, 2, 3, 4 \text{ and } x_0 = x_5 = 0.$$

Solve for the x_i.

One should get the system into the matrix equation

$$\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$$

One could note that the coefficient matrix is symmetric, banded, and toeplitz.